

AR-QUIVER APPROACH TO AFFINE CANONICAL BASIS ELEMENTS

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ABSTRACT. This is the continuation of [Li]. We describe the affine canonical basis elements in the case when the affine quiver has arbitrary orientation. This generalizes the description in [Lus3].

1. INTRODUCTION

Let \mathbf{U}^- be the negative part of the quantized enveloping algebra \mathbf{U} of the Kac-Moody Lie algebra associated to a symmetric generalized Cartan matrix C . One of the milestones in the Lie theory is the discovery of Lusztig's canonical basis \mathbf{B} (or Kashiwara's global crystal basis [K]) of \mathbf{U}^- . The canonical basis \mathbf{B} possesses many remarkable properties such as total positivity and integrality (see [Lus2] and [Lus4]).

There are two different approaches of defining the canonical basis \mathbf{B} . One is algebraic and the other geometric. For the algebraic approach, see [Lus1] and [K]. The geometric approach is also done in [Lus1] when C is positive definite and is extended to all cases in [Lus2]. It is shown in [GL] that the two different approaches produce the same basis in \mathbf{U}^- .

Let Q be a quiver such that the associated Cartan matrix is C . In the geometric approach, Lusztig studies certain perverse sheaves over the representation space $E_{V,Q}$ of Q . An algebra \mathcal{K}_Q is constructed and is shown to be isomorphic to \mathbf{U}^- . The simple perverse sheaves in \mathcal{K}_Q form a basis \mathcal{B}_Q of \mathcal{K}_Q . The canonical basis \mathbf{B} is then defined to be \mathcal{B}_Q if one identifies \mathcal{K}_Q with \mathbf{U}^- . In other words, the geometric approach gives a geometric realization of \mathbf{U}^- and \mathbf{B} for each Q such that the associated Cartan matrix is C . (Note that for each C , there may be several quivers such that the associated Cartan matrices are C .)

It is well-known that Lie theory and the representation theory of quivers have very deep connections ever since Gabriel's theorem [G] was found. The interaction between the two theories attracts a lot of attentions since then. Note that \mathbf{U}^- and \mathbf{B} are objects coming purely from Lie theory. Although the framework of Lusztig's geometric realizations of \mathbf{U}^- and \mathbf{B} is the representation space $E_{V,Q}$ of the quiver Q . No representation theory of quivers is used explicitly. Nor are there clear connections between the representation theory of quivers and Lusztig's geometric realizations of \mathbf{U}^- and \mathbf{B} . One may ask to what extent the representation theory of quivers can help in understanding the quantum group \mathbf{U} and the canonical basis \mathbf{B} or \mathcal{B}_Q . Keeping this in mind, one may ask the following natural questions.

Question 1.1. *Characterize the elements in \mathcal{B}_Q by using the representation theory of quivers. More precisely, describe what the supports and the corresponding local systems of the elements in \mathcal{B}_Q are in terms of representations of quivers.*

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Question 1.2. *Recover the canonical basis via bases arising in Ringel-Hall algebras.*

Question 1.3. *Show the positivity or the integrality of the canonical basis \mathbf{B} in an elementary way with the help of the representation theory of quivers, instead of the theory of perverse sheaves.*

It seems very difficult at this moment to answer Question 1.3. For an attempt to answer Question 1.2, see [Lus1] when Q is of finite type and [LXZ] when Q is of affine type and the references therein. By now there are only partial answers to Question 1.1.

The answer to Question 1.1 is known when Q is of finite type, i.e., when the Cartan matrix is positive definite. The elements in \mathcal{B}_Q are simple perverse sheaves whose supports are G_V -orbits O in $E_{V,Q}$ and whose restrictions to O are the constant sheaf on O . The answer to Question 1.1 is known when Q is a cyclic quiver. Elements in \mathcal{B}_Q are simple perverse sheaves whose supports are the aperiodic G_V -orbits O in $E_{V,Q}$ and whose restrictions to O are the constant sheaf on O .

From now on in the introduction, we assume that Q is an affine quiver but not a cyclic quiver.

When Q is a McKay quiver, i.e., all vertices are either a sink or a source, the answer to Question 1.1 is given in [Lus3, Theorem 6.16] by using the representation theory of McKay quivers. The theory of representation of McKay quivers in [Lus3] is based on McKay's correspondence. In [Lus3], only McKay quivers and cyclic quivers are considered. Note that the construction of \mathcal{K}_Q and \mathcal{B}_Q applies to any quivers and the language used in [DR] for the representation theory of affine quivers works for all affine quivers, not just McKay quivers.

Given any two affine quivers Q and Q' such that the associated generalized Cartan matrices coincide. Although the inverse images $\phi^{-1}(\mathcal{B}_Q)$ and $(\phi')^{-1}(\mathcal{B}_{Q'})$ coincide (see Theorem 4.11), the isomorphism classes of simple equivariant perverse sheaves in \mathcal{B}_Q and $\mathcal{B}_{Q'}$ may be quite different. This becomes obvious when Q and Q' are the Kronecker quiver and the cyclic quiver with two vertices, respectively.

The goal of this paper is to give a complete answer to Question 1.1 for any affine quiver, by carrying out Lusztig's arguments in [Lus3] to any affine quiver via the representation theory developed in [DR].

In [Li], such a goal has been accomplished when Q is of type \tilde{A} . Similar to [Li], the crucial part on the way of generalizing Lusztig's argument is still to construct a 'nice' functor from the category of nilpotent representations of a cyclic quiver C_p to the full subcategory $\text{Rep}(T)$ generated by a tube T of period p which will give an equivariant morphisms between the representation varieties and transporting the equivariant simple perverse sheaves to equivariant simple perverse sheaves. It turns out that the Hall functor in [FMV] will suffice to overcome this difficulty. This functor gives us all the properties we need in the proof of Lemma 5.8 in Section 5.5.

It should be mentioned that after the preprint of the paper is written, the second author received a preprint [N] by Nakajima, in which he outlines an approach in answering Question 1.1 when the quivers are of type \tilde{D} and \tilde{E} by using the description of \mathcal{B}_Q for McKay quivers by Lusztig ([Lus3] Theorem 6.16) and then apply the reflection functors. Our argument does not depend on Theorem 6.16 in [Lus3].

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discussing questions in his paper [N]. The first author thanks the second author for allowing him to use the results in this paper as part of his Ph.D. thesis.

2. REPRESENTATION THEORY OF AFFINE QUIVERS

In this section, we give a brief review of representation theory of affine quivers. See [DR] and [BGP] for more details.

2.1. Preliminary. A *quiver* is an oriented graph. It is a quadruple $Q = (I, \Omega, h, t)$, where I and Ω are two finite sets and h, t are two maps from Ω to I such that $h(\omega) \neq t(\omega)$ for any $\omega \in \Omega$. I and Ω are called the vertex and arrow sets respectively. Pictorially, $t(\omega) \xrightarrow{\omega} h(\omega)$ stands for any arrow $\omega \in \Omega$ and we call $h(\omega)$ and $t(\omega)$ the head and the tail of the arrow ω respectively. An *affine quiver* is a quiver whose underlying graph is of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or \tilde{E}_8 .

Throughout this paper, all quivers considered will be affine. We fix an algebraically closed field K once and for all.

A *representaion* of a quiver over K is a pair (V, x) , where $V = \bigoplus_{i \in I} V_i$ is an I -graded finite dimensional K -vector space and $x = \{x_\omega : V_{t(\omega)} \rightarrow V_{h(\omega)} \mid \omega \in \Omega\}$ is a collection of linear maps. A morphism $f : (V, x) \rightarrow (W, y)$ is a collection of linear maps $\{f_i : V_i \rightarrow W_i \mid i \in I\}$ such that $f_{h(\omega)} x_\omega = y_\omega f_{t(\omega)}$, for any $\omega \in \Omega$. This defines a (abelian) category, denoted by $\text{Rep}(Q)$.

A *nilpotent* representation is a representation (V, x) having the property: there is an N such that for any $\omega_1, \dots, \omega_N$ in Ω satisfying $t(\omega_k) = h(\omega_{k-1})$ for any k , the composition of morphisms $x_{\omega_N} \circ \dots \circ x_{\omega_1}$ is zero. Denote by $\text{Nil}(Q)$ the full subcategory of $\text{Rep}(Q)$ of all nilpotent representations of Q . Note that when Q has no oriented cycles, every representation is nilpotent, i.e., $\text{Nil}(Q) = \text{Rep}(Q)$.

Denote by $\text{Ind}(Q)$ the set of representatives of all pairwise nonisomorphic indecomposable representations in $\text{Rep}(Q)$.

The Euler form $<, >$ on $\mathbb{Z}[I]$ is defined by

$$< \alpha, \beta > = \sum_{i \in I} \alpha_i \beta_i - \sum_{\omega \in \Omega} \alpha_{t(\omega)} \beta_{h(\omega)},$$

for any $\alpha = \sum_i \alpha_i i, \beta = \sum_i \beta_i i \in \mathbb{Z}[I]$. The symmetric Euler form $(,)$ on $\mathbb{Z}[I]$ is defined to be

$$(\alpha, \beta) = < \alpha, \beta > + < \beta, \alpha >,$$

for any $\alpha = \sum_i \alpha_i i, \beta = \sum_i \beta_i i \in \mathbb{Z}[I]$. Given $\mathbf{V} = (V, x) \in \text{Rep}(Q)$, denote by $|\mathbf{V}|$ its dimension vector $\sum_{i \in I} (\dim V_i) i \in \mathbb{Z}[I]$. For any $M, N \in \text{Rep}(Q)$, we have from [DR]

$$< |M|, |N| > = \dim \text{Hom}_Q(M, N) - \dim \text{Ext}_Q^1(M, N).$$

Given any I -graded K -vector space V , let

$$E_{V, \Omega} = \bigoplus_{\omega \in \Omega} \text{Hom}(V_{t(\omega)}, V_{h(\omega)}) \quad \text{and} \quad G_V = \prod_{i \in I} \text{GL}(V_i),$$

where $\text{GL}(V_i)$ is the general linear group of V_i for all $i \in I$. G_V acts on $E_{V, \Omega}$ naturally, i.e., $g.x = y$, where $y_\omega = g_{h(\omega)} x_\omega g_{t(\omega)}^{-1}$ for all $\omega \in \Omega$. Note that for any $x \in E_{V, \Omega}$, (V, x) is a representation of Q . We call that x is nilpotent if (V, x) is a nilpotent representation.

2.2. BGP reflection functors. A vertex $i \in I$ is called a sink (resp., a source) if $i \in \{t(\omega), h(\omega)\}$ implies $h(\omega) = i$ (resp., $t(\omega) = i$) for any $\omega \in \Omega$.

For any $i \in I$, let $\sigma_i Q = (I, \Omega, h', t')$ be the quiver with the orientation (t', h') defined by

$$\begin{aligned} t'(\omega) &= t(\omega) \text{ and } h'(\omega) = h(\omega) \text{ if } i \notin \{t(\omega), h(\omega)\}; \\ t'(\omega) &= h(\omega) \text{ and } h'(\omega) = t(\omega) \text{ if } i \in \{t(\omega), h(\omega)\}; \end{aligned}$$

for any $\omega \in \Omega$. In other words, $\sigma_i Q$ is the quiver obtained by reversing the arrows in Q that start or terminate at i .

If i is a sink in Q , define a functor

$$\Phi_i^+ : \text{Rep}(Q) \rightarrow \text{Rep}(\sigma_i Q)$$

in the following way. For any $(V, x) \in \text{Rep}(Q)$, $\Phi_i^+(V, x) = (W, y) \in \text{Rep}(\sigma_i Q)$, where $W = \bigoplus_{j \in I} W_j$ such that

$$W_j = V_j, \text{ if } j \neq i;$$

$$W_i \text{ is the kernel of the linear map } \sum_{\omega \in \Omega: h(\omega)=i} x_\omega : \bigoplus_{\omega \in \Omega: h(\omega)=i} V_{t(\omega)} \rightarrow V_i;$$

and $y = (y_\omega \mid \omega \in \Omega)$ such that

$$y_\omega = x_\omega, \text{ if } t'(\omega) \neq i;$$

$$y_\omega : W_i \rightarrow W_{h'(\omega)} \text{ is the composition of the natural maps:}$$

$$W_i \rightarrow \bigoplus_{\omega \in \Omega: h(\omega)=i} V_{t(\omega)} = \bigoplus_{\omega \in \Omega: t'(\omega)=i} W_{h'(\omega)} \rightarrow W_{h'(\omega)},$$

$$\text{if } t'(\omega) = i.$$

Note that the assignments extend to a functor. Φ_i^+ is called the BGP reflection functor with respect to the sink i .

Since Q has no oriented cycles, we can order the vertices in I , say (i_1, \dots, i_n) ($|I| = n$), in such a way that i_r is a sink in the quiver $\sigma_{i_{r-1}} \cdots \sigma_{i_1} Q$. Then from [BGP],

$$Q = \sigma_{i_n} \cdots \sigma_{i_1} Q.$$

Define the Coxeter functor $\Phi^+ : \text{Rep}(Q) \rightarrow \text{Rep}(Q)$ to be

$$\Phi^+ = \Phi_{i_n}^+ \circ \cdots \circ \Phi_{i_1}^+.$$

Similarly, if i is a source in Q . For any $(V, x) \in \text{Rep}(Q)$, let $\Phi_i^-(V, x) = (W, y)$ be a representation of $\sigma_i Q$, where $W_j = V_j$ if $j \neq i$ and W_i equals the cokernel of the linear map $\sum_{\omega \in \Omega: t(\omega)=i} x_\omega : V_i \rightarrow \bigoplus_{\omega \in \Omega: t(\omega)=i} V_{h(\omega)}$; for any $\omega \in \Omega$, $y_\omega = x_\omega$ if $h'(\omega) \neq i$, otherwise if $h'(\omega) = i$, y_ω is the composition of the natural maps $W_{t'(\omega)} \rightarrow \bigoplus_{\omega \in \Omega: t(\omega)=i} V_{h(\omega)} \rightarrow W_i$. This extends to a functor

$$\Phi_i^- : \text{Rep}(Q) \rightarrow \text{Rep}(\sigma_i Q).$$

The Coxeter functor $\Phi^- : \text{Rep}(Q) \rightarrow \text{Rep}(Q)$ is defined to be

$$\Phi^- = \Phi_{i_1}^- \circ \cdots \circ \Phi_{i_n}^-.$$

2.3. Classification of indecomposable representations. Assume that Q has no oriented cycles. By using the two Coxeter functors Φ^+ and Φ^- , the representations $M \in \text{Ind}(Q)$ are classified into four classes: preprojective, inhomogeneous regular, homogeneous regular and preinjective. More precisely, a representation $M \in \text{Ind}(Q)$ is called preprojective if $(\Phi^+)^r M = 0$, for $r \gg 0$; inhomogeneous regular if $(\Phi^+)^r M \simeq M$ for some $r \geq 2$; homogeneous regular if $(\Phi^+)^r M \simeq M$ for any positive integer r ; and preinjective if $(\Phi^-)^r M = 0$, for $r \gg 0$. In general, $M \in \text{Rep}(Q)$ is preprojective (resp. inhomogeneous regular, homogeneous

regular, preinjective) if all indecomposable direct summands of M are preprojective (resp. inhomogeneous regular, homogeneous regular, preinjective).

Let S_i be the simple representation corresponding to the vertex i . It's a representation (V, x) such that $V_i = k$, $V_j = 0$ if $j \neq i$ and all linear maps x_ω are 0. Note that given a graph, the definition of the simple representation works for any orientation of the graph. By abuse of notation, we always denote by S_i the simple representation corresponding to the vertex i regardless of the orientation.

Fix a sequence (i_1, \dots, i_n) such that i_r is a sink of the quiver $\sigma_{i_{r-1}} \cdots \sigma_{i_1}(Q)$. Let

$$P(i_r) = \Phi_{i_1}^- \circ \cdots \circ \Phi_{i_{r-1}}^-(S_{i_r}) \quad \text{and} \quad I(i_r) = \Phi_{i_n}^+ \circ \cdots \circ \Phi_{i_{r+1}}^+(S_{i_r}).$$

Then we have

- (1) $M \in \text{Ind}(Q)$ is projective iff $M \simeq P(i_r)$.
- (1') $M \in \text{Ind}(Q)$ is injective iff $M \simeq I(i_r)$.
- (2) $M \in \text{Ind}(Q)$ is preprojective iff $M = (\Phi^-)^r P(i)$.
- (3) $M \in \text{Ind}(Q)$ is preinjective iff $M = (\Phi^+)^r I(i)$.

Let $\text{Reg}(Q)$ be the full subcategory of $\text{Rep}(Q)$ whose objects are inhomogeneous regular and homogeneous representations. Then we have

- (4) $\text{Reg}(Q)$ is an extension-closed full subcategory of $\text{Rep}(Q)$.
- (5) Φ^+ is an autoequivalence on $\text{Reg}(Q)$. Φ^- is its inverse.

The simple objects in $\text{Reg}(Q)$ are called regular simple representations. For each regular simple representation R , there exists a positive integer r such that $(\Phi^+)^r R = R$. We call the smallest one, p , the period of R under Φ^+ . The set $\{R, \dots, (\Phi^+)^{p-1} R\}$ is called the Φ^+ -orbit of R . Given a Φ^+ -orbit of a regular simple representation, the corresponding *tube*, say T , is a set of isoclasses of all indecomposable regular representations whose regular composition factors belong to this orbit. Let $\text{Rep}(T)$ be the full subcategory of $\text{Rep}(Q)$ the objects of which are direct sums of indecomposable representations in T . We have the following facts:

- (6) Every regular indecomposable representation belongs to a unique tube;
- (7) Every indecomposable object in a tube has the same period under Φ^+ ;
- (8) All but finitely many regular simple objects have period one;

Let p be the cardinality of the set of isomorphism classes of simple objects in $\text{Rep}(T)$ and p_T be the period of T . Then

$$(9) \quad p = p_T$$

Given any representation $M \in \text{Rep}(T)$ with T of period p , M is called aperiodic if for any $N \in T$ not all the indecomposable representations

$$N, (\Phi^+) N, \dots, (\Phi^+)^{p-1} N$$

are direct summands of M . Given any $x \in E_{V,\Omega}$ such that the representation (V, x) is aperiodic, we call the G_V -orbit O_x of x aperiodic.

The following lemma will be needed in the proof of Proposition 5.10.

Lemma 2.4. *Let M and $N \in \text{Ind}(Q)$ be one of the following cases.*

- (1) $M = (\Phi^+)^m I(i_r)$ and $N = (\Phi^+)^n I(i_{r'})$, for $m > n$ or $m = n$ and $r \leq r'$.
- (2) $M = (\Phi^-)^m P(i_r)$ and $N = (\Phi^-)^n P(i_{r'})$, for $m < n$ or $m = n$ and $r \leq r'$.
- (3) M and N are both regular, but they are not in the same tube.
- (4) M is nonpreinjective and N is preinjective.
- (5) M is preprojective and N is nonpreprojective.

Then we have

- (i) $\text{Ext}^1(M, N) = 0$;
- (ii) $\text{Hom}(N, M) = 0$; if M and N are not isomorphic.

The Lemma follows from [CB, Chapter 6, Lemma 1].

2.5. Tubes in a noncyclic quiver. Let $Q = (I, \Omega; s, t)$ be an affine quiver other than C_p . Let $R = (\oplus_{i \in I} R_i, r)$ be an inhomogeneous regular simple in $\text{Ind}(Q)$. Let T be a tube in $\text{Rep}(Q)$, consisting of all indecomposable regular representations such that there exists a filtration of subrepresentations whose consecutive quotients are isomorphic to regular simple representations in the Φ^+ -orbit of R . Let $\text{Rep}(T)$ be the full subcategory of $\text{Rep}(Q)$ consisting of direct sums of indecomposable representations in T . The Φ^+ -orbit of R forms a complete list of simple objects in the subcategory $\text{Rep}(T)$. Let p be the minimal $k \geq 1$ such that $(\Phi^+)^k(R) \cong R$. For convenience, we write $R_z = (\oplus_{i \in I} R_{z,i}, r_z)$ for $(\Phi^+)^k(R)$ if $[k] = z$ in $\mathbb{Z}/p\mathbb{Z}$, where $[k]$ is the class of k in $\mathbb{Z}/p\mathbb{Z}$. Note that $R_{z,i}$ is a vector space and $r_z = (r_{z,\omega})$ with $r_{z,\omega} : R_{z,t(\omega)} \rightarrow R_{z,h(\omega)}$ is a linear transformation for each $\omega \in \Omega$. These regular simples in T satisfy the following properties.

- (1) $\text{Hom}_Q(R_z, R_z) = K$ and $\text{Hom}_Q(R_z, R_{z'}) = 0$ if $z \neq z'$;
- (2) $\text{Ext}_Q^1(R_z, R_{z-1}) = K$ and $\text{Ext}_Q^1(R_z, R_{z'}) = 0$ if $z \neq z' + 1$;
- (3) All higher extension groups of the simple objects in $\text{Rep}(T)$ vanish.

For each $z \in \mathbb{Z}/p\mathbb{Z}$, fix an extension $E_z = (\oplus_{i \in I} E_{z,i}, e_z)$ of R_z by R_{z-1} such that the exact sequence $0 \rightarrow R_{z-1} \rightarrow E_z \rightarrow R_z \rightarrow 0$ is nonsplit. Notice that $E_{z,i} = R_{z-1,i} \oplus R_{z,i}$ for any $i \in I$. (In another word, we fix a basis of the extension group $\text{Ext}_Q^1(R_z, R_{z-1})$.) For each arrow $\omega \in \Omega$, the linear map $e_{z,\omega} : E_{z,t(\omega)} \rightarrow E_{z,h(\omega)}$ induces a linear map

$$(4) \quad l_{z,\omega} : R_{z,t(\omega)} \rightarrow R_{z-1,h(\omega)}.$$

2.6. Cyclic quivers. Let C_p be a cyclic quiver of p vertices. More precisely, C_p is defined to be the quadruple $C_p = (I_p, \Omega_p; s_p, t_p)$, where

- (i) $I_p = \mathbb{Z}/p\mathbb{Z}$,
- (ii) $\Omega_p = \{\omega_z \mid z \in \mathbb{Z}/p\mathbb{Z}\}$,
- (iii) $t_p(\omega_z) = z$ and $h_p(\omega_z) = z - 1$ for any $z \in \mathbb{Z}/p\mathbb{Z}$.

We sometimes write $z \rightarrow z - 1$ for the arrow ω_z . For each vertex $z \in \mathbb{Z}/p\mathbb{Z}$, denote by s_z the corresponding simple representation, i.e., s_z is the representation whose associated vector space is K at vertex z and 0 elsewhere and whose associated linear maps are zero. For each $\lambda \in K^*$, define a representation t_λ by associating the one dimensional vector space K to each vertex and the identity map to each arrow except the arrow $0 \rightarrow p - 1$ which is associated with the scalar map λId , where Id is the identity map. The union $\{s_z \mid z \in \mathbb{Z}/p\mathbb{Z}\} \cup \{t_\lambda \mid \lambda \in K^*\}$ forms a complete list of pairwise nonisomorphic simple representations in $\text{Rep}(C_p)$. In

particular, the subset $\{s_z \mid z \in \mathbb{Z}/p\mathbb{Z}\}$ is a complete list of pairwise nonisomorphic simple objects in $\text{Nil}(C_p)$. Moreover, they satisfy the following properties.

- (1) $\text{Hom}_{C_p}(s_z, s_z) = K$ and $\text{Hom}_{C_p}(s_z, s_{z'}) = 0$ if $z \neq z'$;
- (2) $\text{Ext}_{C_p}^1(s_z, s_{z-1}) = K$ and $\text{Ext}_{C_p}^1(s_z, s_{z'}) = 0$ if $z' \neq z - 1$;
- (3) $\text{Hom}_{C_p}(t_\lambda, t_\lambda) = K$ and $\text{Hom}_{C_p}(t_\lambda, t_{\lambda'}) = 0$ if $\lambda \neq \lambda'$;
- (4) $\text{Ext}_{C_p}^1(t_\lambda, t_\lambda) = K$ and $\text{Ext}_{C_p}^1(t_\lambda, t_{\lambda'}) = 0$ if $\lambda \neq \lambda'$;
- (5) $\text{Hom}_{C_p}(s_z, t_\lambda) = 0$ and $\text{Hom}_{C_p}(t_\lambda, s_z) = 0$ for any $z \in \mathbb{Z}/p\mathbb{Z}$ and $\lambda \in K^*$;
- (6) $\text{Ext}_{C_p}^1(s_z, t_\lambda) = 0$ and $\text{Ext}_{C_p}^1(t_\lambda, s_z) = 0$ for any $z \in \mathbb{Z}/p\mathbb{Z}$ and $\lambda \in K^*$;
- (7) All higher extension groups of the simple objects vanish.

Let $s_{z,l}$ be the indecomposable representation of C_p such that its socle is s_i and its length is l for $z \in I_p$ and $l \in \mathbb{N}$. Given any representation M in $\text{Rep}(C_p)$, M is called aperiodic if for each $l \in \mathbb{N}$ not all of the indecomposable representations

$$s_{0,l}, s_{1,l}, \dots, s_{p-1,l}$$

are direct summands of M .

Given any $x \in E_{V, C_p}$, if the representation (V, x) is aperiodic, then we call the G_V -orbit O_x of x aperiodic.

3. HALL FUNCTORS AND HALL MORPHISMS

3.1. The Hall functor F . Let T be a tube (Section 2.5) in $\text{Rep}(Q)$. Recall that $R_z = (\oplus_{i \in I} R_{z,i}, r_z)$ ($z \in \mathbb{Z}/p\mathbb{Z}$) are the pairwise nonisomorphic simple objects in $\text{Rep}(T)$. $l_{z,\omega} : R_{z,t(\omega)} \rightarrow R_{z-1,h(\omega)}$ ($\omega \in \Omega$) is a linear map defined Section 2.5.

For any representation $(V, \theta) \in \text{Rep}(C_p)$ (Section 2.6), define a representation

$$F(V, \theta) = (F(V), F(\theta)) \in \text{Rep}(Q)$$

by

- (i) $F(V)_i = \oplus_{z \in \mathbb{Z}/p\mathbb{Z}} V_z \otimes R_{z,i}$ for any $i \in I$,
- (ii) $F(\theta)_\omega = \sum_{z \in \mathbb{Z}/p\mathbb{Z}} (I_z \otimes r_{z,\omega} + \theta_{z \rightarrow z-1} \otimes l_{z,\omega})$, for each arrow $\omega \in \Omega$.

(Here $I_z : V_z \rightarrow V_z$ in (ii) is the identity map.) Note that $F(\theta)_\omega$ is a linear map for $F(V)_{t(\omega)}$ to $F(V)_{h(\omega)}$. This map extends to a functor $F : \text{Rep}(C_p) \rightarrow \text{Rep}(Q)$. By construction, we have

Lemma 3.2. (a) F is an exact functor.

- (b) $F(s_z) = R_z$ for all $z \in \mathbb{Z}/p\mathbb{Z}$.
- (c) $F(V, \theta) \in \text{Rep}(T)$ for any $(V, \theta) \in \text{Nil}(C_p)$.
- (d) $F(t_\lambda)$ is a homogeneous regular simple for any $\lambda \in K^*$.

Proof. From the construction, F is exact on each vector space. To show (a), one only needs to check that F is a functor which is straightforward verified. (b) can be checked directly.

For any $(V, \theta) \in \text{Nil}(C_p)$, it is well-known that there exists a sequence of subrepresentations

$$(V, \theta) \supseteq (V^1, \theta^1) \supseteq \dots \supseteq (V^n, \theta^n) = 0,$$

such that $(V^l, \theta^l)/(V^{l+1}, \theta^{l+1})$ are simple representations of type s_1, \dots, s_p . From this and the construction of F , there is a sequence of subrepresentations of $F(V, \theta)$:

$$F(V, \theta) \supseteq F(V^1, \theta^1) \supseteq \dots \supseteq F(V^n, \theta^n) = 0,$$

satisfying $F(\mathbb{V}^l, \theta^l)/F(\mathbb{V}^{l+1}, \theta^{l+1}) \simeq R_z$ by (a) for some $z \in \mathbb{Z}/p\mathbb{Z}$. Therefore, (c) follows from (b).

To prove (d), when Q is of type \tilde{A}_n , the functor F coincides with the functor G in [Li]. From the construction of G , $F(t_\lambda)$ is a homogeneous regular simple. Furthermore, $\{F(t_\lambda) | \lambda \in K^*\}$ is a complete list of pairwise nonisomorphic homogeneous regular simples in $\text{Rep}(Q)$. When Q is of type \tilde{D}_n or \tilde{E}_m ($m = 6, 7$ and 8), we choose the orientation Ω' given in [DR, P. 46-49]. We denote by Q' the corresponding quiver. By direct computation, $F(t_\lambda)$ is a homogeneous regular simple. Furthermore, $\{F(t_\lambda) | \lambda \in K^*\}$ is a complete list of pairwise nonisomorphic homogeneous regular simples in $\text{Rep}(Q')$. On the other hand, by [BGP], there exists a sequence i_1, \dots, i_k of vertices in I such that it is $(+)$ -accessible with respect to Q and $\sigma_{i_k} \sigma_{i_{k-1}} \cdots \sigma_{i_1} Q = Q'$. Denote by $\Psi^+ : \text{Rep}(Q) \rightarrow \text{Rep}(Q')$ the composition of the corresponding reflection functors Φ_i^+ . We write R'_z for $\Psi^+(R_z)$, for any $z \in \mathbb{Z}/p\mathbb{Z}$. Denote by T' the tube in $\text{Rep}(Q')$ generated by R'_z , for all $z \in \mathbb{Z}/p\mathbb{Z}$. Define the functor $F' : \text{Rep}(C_p) \rightarrow \text{Rep}(Q')$ as the functor F by replacing R_z by R'_z . We then have

$$F' \simeq \Psi^+ \circ F.$$

In fact, it suffices to prove the case when $\sigma_i Q = Q'$. In this case it can be verified directly by the construction of the functors F , F' and Φ_i^+ .

Note that Ψ^+ sends homogeneous regular simples to homogeneous regular simples. This proves (d). \square

We denote by HT the full subcategory of $\text{Rep}(Q)$ generated by R_z and $F(t_\lambda)$, for all $z \in \mathbb{Z}/p\mathbb{Z}$, $\lambda \in K^*$. From the above Lemma, F induces a functor from $\text{Rep}(C_p)$ to HT, still denoted by F . We have

Lemma 3.3. *The induced functor $F : \text{Rep}(C_p) \rightarrow \text{HT}$ is a categorical equivalence.*

Proof. By Lemma 2.5 (a), F induces maps:

- (1) $F_z : \text{Ext}_{C_p}^1(s_z, s_{z-1}) \rightarrow \text{Ext}_Q^1(F(s_z), F(s_{z-1}))$ for any $z \in \mathbb{Z}/p\mathbb{Z}$.
- (2) $F_\lambda : \text{Ext}_{C_p}^1(t_\lambda, t_\lambda) \rightarrow \text{Ext}_Q^1(F(t_\lambda), F(t_\lambda))$ for any $\lambda \in K^*$.

They are all injective and K -linear. So F_z and F_λ are bijective, for any $z \in \mathbb{Z}/p\mathbb{Z}$ and $\lambda \in K^*$. We then have the equivalence by the following Lemma stated in [GRK, P. 129].

Lemma 3.4. *Let $E : \mathcal{B} \rightarrow \mathcal{C}$ be an exact functor between two abelian aggregates whose objects have finite Jordan-Hölder series. Then E is an equivalence if and only if the following two conditions are satisfied:*

- (i) *E maps simples onto simples and induces a bijection between their sets of isoclasses.*
- (ii) *For all simples $S, T \in \mathcal{B}$, the map $\text{Ext}_{\mathcal{B}}^i(S, T) \rightarrow \text{Ext}_{\mathcal{C}}^i(ES, ET)$ induced by E is bijective for $i = 1$ and injective for $i = 2$.*

\square

Remark. The functor F is a Hall functor in [FMV]. A prototype of the functor F can be found in [GRK]. The restriction $F : \text{Nil}(C_p) \rightarrow \text{Rep}(T)$ is equivalent.

3.5. The Hall morphism F . The assignment $(\mathbb{V}, \theta) \mapsto (F(\mathbb{V}), F(\theta))$ in the above subsection gives a morphism of varieties:

$$(a) \quad F : E_{\mathbb{V}, \Omega_p} \rightarrow E_{F(\mathbb{V}), \Omega}, \quad \theta \mapsto F(\theta).$$

Note that both $E_{\mathbb{V}, \Omega_p}$ and $E_{F(\mathbb{V}), \Omega}$ are K -vector spaces and the map is affine linear (a linear map plus a constant map).

We set $V = F(\mathbb{V})$. By definition, V is an I -graded K -vector space. Denote by E_1 the image of $E_{\mathbb{V}, \Omega_p}$ under F . (Note that E_1 is a translate of a vector subspace of $E_{V, \Omega}$.) Denote by E_2 the set of all elements $x \in E_{V, \Omega}$ such that x is in the same G_V -orbit of some element in E_1 . Clearly,

$$(b) \quad E_1 \subseteq E_2 \quad \text{and} \quad E_2 \quad \text{is a } G_V\text{-stable subvariety of } E_{V, \Omega}.$$

Moreover, the assignment $(\mathbb{V}, \theta) \mapsto (F(\mathbb{V}), F(\theta))$ induces an algebraic group homomorphism defined by

$$(c) \quad F : G_{\mathbb{V}} \rightarrow G_V, \quad g \mapsto g \otimes 1, \quad \text{for any } g \in G_{\mathbb{V}},$$

where $(g \otimes 1)_i = \bigoplus_{z \in \mathbb{Z}/p\mathbb{Z}} g_z \otimes I_{z,i} : \bigoplus_z \mathbb{V}_z \otimes R_{z,i} \rightarrow \bigoplus_z \mathbb{V}_z \otimes R_{z,i}$ for all $i \in I$, with $I_{z,i} : R_{z,i} \rightarrow R_{z,i}$ being the identity map. For simplicity, we write H_V for $F(G_{\mathbb{V}})$. It is an algebraic subgroup of G_V . Note that $G_{\mathbb{V}} \simeq H_V$. Also there is an action of H_V on E_1 induced by the action of $G_{\mathbb{V}}$ on $E_{\mathbb{V}, \Omega_p}$. By definitions, the action of $G_{\mathbb{V}}$ on $E_{\mathbb{V}, \Omega_p}$ is compatible with the action of H_V on E_1 , i.e.,

$$(d) \quad F(g\theta) = F(g)F(\theta), \quad \text{for any } g \in G_{\mathbb{V}} \quad \text{and} \quad \theta \in E_{\mathbb{V}, \Omega_p}.$$

Furthermore, we have

$$(e) \quad \text{The map } F : E_{\mathbb{V}, \Omega_p} \rightarrow E_1 \text{ is an isomorphism of varieties.}$$

$$(f) \quad \text{For } x, x_1 \in E_1, \{ \xi \in G_V \mid \xi x = x_1 \} \subseteq H_V. \text{ In particular, } H_V = \text{Stab}_{G_V}(F(0)).$$

In fact, $x = F(\theta)$ and $x_1 = F(\theta_1)$, for some $\theta, \theta_1 \in E_{\mathbb{V}, \Omega_p}$. So $\xi F(\theta) = F(\theta_1)$. That is $\xi \in \text{Hom}_Q((V, F(\theta)), (V, F(\theta_1)))$. Since the functor $F : \text{Rep}(C_p) \rightarrow \text{HT}$ is a categorical equivalence, it is then fully faithful. So $\text{Hom}_Q((V, F(\theta)), (V, F(\theta_1))) = F(\text{Hom}_{C_p}((\mathbb{V}, \theta), (\mathbb{V}, \theta_1)))$. Therefore, $\xi = F(g)$ for some $g \in G_{\mathbb{V}}$.

Let

$$\text{GL}(V_i)^0 = \prod_{z \in \mathbb{Z}/\mathbb{Z}_p} \text{GL}(\mathbb{V}_z \otimes R_{z,i}) \quad \text{and} \quad G_V^0 = \prod_{i \in I} \text{GL}(V_i)^0.$$

Then G_V^0 is a Levi subgroup of the reductive group G_V . Let

$$\begin{aligned} E_{V, \Omega}^0 &= \bigoplus_{\omega \in \Omega} \bigoplus_{z \in \mathbb{Z}/\mathbb{Z}_p} \text{Hom}_K(\mathbb{V}_z \otimes R_{z, t(\omega)}, \mathbb{V}_z \otimes R_{z, h(\omega)}), \\ E_{V, \Omega}^1 &= \bigoplus_{\omega \in \Omega} \bigoplus_{z \in \mathbb{Z}/\mathbb{Z}_p} \text{Hom}_K(\mathbb{V}_z \otimes R_{z, t(\omega)}, \mathbb{V}_{z-1} \otimes R_{z-1, h(\omega)}). \end{aligned}$$

By definition

$$G_V^0 E_1 \subseteq E_{V, \Omega}^0 \oplus E_{V, \Omega}^1.$$

The map $\phi : G_V^0 E_1 \rightarrow G_V^0 F(0)$ ($gF(\theta) \mapsto gF(0)$) is then a restriction of the second projection $E_{V, \Omega}^0 \oplus E_{V, \Omega}^1 \rightarrow E_{V, \Omega}^0$. Hence ϕ is a morphism of varieties. By [S, Lemma 4 in 3.7] and the fact that $G_V^0/H_V \simeq G_V^0 F(0)$, we have

$$(g) \quad G_V^0 \times^{H_V} E_1 \simeq G_V^0 E_1.$$

Define an action of H_V on $G_V \times E_1$ by $(\zeta, x) \cdot \xi = (\zeta \xi, \xi^{-1}x)$ for any $(\zeta, x) \in G_V \times E_1$ and $\xi \in H_V$. Define a map $\tau : G_V \times E_1 \rightarrow E_2$ by $(\zeta, x) \mapsto \zeta \cdot x$ for any $(\zeta, x) \in G_V \times E_1$. By (j), the morphism τ is an H_V -orbit map, i.e., all the fibres of τ over $x \in E_2$ are H_V -orbit. Thus it induces a bijective morphism of varieties

$$\bar{\tau} : G_V \times^{H_V} E_1 \rightarrow E_2.$$

Note that $H_V \subseteq G_V^0$. By [S] and (g), we have

$$G_V \times^{H_V} E_1 \simeq G_V \times^{G_V^0} (G_V^0 \times^{H_V} E_1) \simeq G_V \times^{G_V^0} (G_V^0 E_1).$$

Thus $\bar{\tau}$ induces a bijective morphism of varieties

$$(h) \quad \bar{\tau} : G_V \times^{G_V^0} (G_V^0 E_1) \rightarrow E_2.$$

Let U^+ (resp. U^-) be the block upper (resp. lower) triangle matrices with respect to G_V^0 in G_V . U^+ and U^- are unipotent radicals of opposite parabolics with G_V^0 as Levi subgroup. Then $U^+ \times G_V^0 \times U^-$ is an affine open subvariety in G_V . Thus $U := U^+ \times U^- \times G_V^0 E_1$ is an affine open subvariety in $G_V \times^{G_V^0} G_V^0 E_1$. By restricting to the affine open subvariety U , the morphism $\bar{\tau}$ becomes an isomorphism onto its image by direct matrix computations (the inverse of $\bar{\tau}$ is algebraic). Now that $\{gU \mid g \in G_V\}$ is an open cover of $G_V \times^{G_V^0} (G_V^0 E_1)$, we have

Lemma 3.6. $G_V \times^{H_V} E_1 \simeq E_2$ as G_V -varieties.

In particular,

$$(i) \quad O_x = G_V \times^{H_V} O'_x,$$

where $x \in E_1$, O_x is the G_V -orbit of x in E_2 and O'_x is the H_V -orbit of x in E_1 .

4. THE CANONICAL BASIS

In this section, we recall Lusztig's geometric realization of the canonical basis of \mathbf{U}^- .

4.1. Notations. We fix some notations, most of them are consistent with the notations in [Lus4].

Fix a prime l that is invertible in K . Given any algebraic variety X over K , denote by $\mathcal{D}_c^b(X)$ the bounded derived category of complexes of l -adic sheaves on X ([BBD]). Let $\mathcal{M}(X)$ be the full subcategory of $\mathcal{D}_c^b(X)$ consisting of all perverse sheaves on X ([BBD]).

Let G be a connected algebraic group. Assume that G acts on X algebraically. Denote by $\mathcal{D}_G(X)$ the full subcategory of $\mathcal{D}_c^b(X)$ consisting of all G -equivariant complexes over X . Similarly, denote by $\mathcal{M}_G(X)$ the full subcategory of $\mathcal{M}(X)$ consisting of all G -equivariant perverse sheaves ([Lus4]).

Let $\bar{\mathbb{Q}}_l$ be an algebraic closure of the field of l -adic numbers. By abuse of notation, denote by $\bar{\mathbb{Q}}_l = (\bar{\mathbb{Q}}_l)_X$ the complex concentrated on degree zero, corresponding to the constant l -adic sheaf over X . For any complex $K \in \mathcal{D}_c^b(X)$ and $n \in \mathbb{Z}$, let $K[n]$ be the complex such that $K[n]^i = K^{n+i}$ and the differential is multiplied by a factor $(-1)^n$. Denote by $\mathcal{M}(X)[n]$ the full subcategory of $\mathcal{D}_c^b(X)$ objects of which are of the form $K[n]$ with $K \in \mathcal{M}(X)$. For any $K \in \mathcal{D}_c^b(X)$ and $L \in \mathcal{D}_c^b(Y)$, denote by $K \boxtimes L$ the external tensor product of K and L in $\mathcal{D}_c^b(X \times Y)$.

Let $f : X \rightarrow Y$ be a morphism of varieties, denote by $f^* : \mathcal{D}_c^b(Y) \rightarrow \mathcal{D}_c^b(X)$ and $f_! : \mathcal{D}_c^b(X) \rightarrow \mathcal{D}_c^b(Y)$ the inverse image functor and the direct image functor with compact support, respectively.

If G acts on X algebraically and f is a principal G -bundle, then f^* induces a functor (still denote by f^*) of equivalence between $\mathcal{M}(Y)[\dim G]$ and $\mathcal{M}_G(X)$. Its inverse functor is denoted by $f_b : \mathcal{M}_G(X) \rightarrow \mathcal{M}(Y)[\dim G]$ ([Lus4]).

4.2. Lusztig's induction functor. We recall the (geometric) definition of the canonical basis from [Lus2] (or [Lus4]). Let $V = \bigoplus_{i \in I} V_i$ be an I -graded K -vector space. We have the following data:

- (1) The set $\mathcal{X}_{|V|}$. It consists of all sequences $\nu = (\nu^1, \dots, \nu^n)$ such that $\sum_{m=1}^n \nu^m = |V|$ and $\nu_{s(\omega)}^m \cdot \nu_{t(\omega)}^m = 0$, for any $\omega \in \Omega$ and $1 \leq m \leq n$.
- (2) The variety \mathcal{F}_ν , for any $\nu = (\nu^1, \dots, \nu^n) \in \mathcal{X}_{|V|}$. It consists of all sequences, $(V = V^0 \supseteq V^1 \supseteq \dots \supseteq V^n = 0)$, of I -graded subspaces of V such that $|V^m/V^{m+1}| = \nu^{m+1}$, for $0 \leq m \leq n-1$.
- (3) The variety $\tilde{\mathcal{F}}_\nu$, for any $\nu = (\nu^1, \dots, \nu^n) \in \mathcal{X}_{|V|}$. It consists of all pairs (x, \mathbf{f}) , where $x \in E_{V,\Omega}$ and $\mathbf{f} \in \mathcal{F}_\nu$, such that \mathbf{f} is x -stable. (Here \mathbf{f} is x -stable means that for any vector subspace, V^m in \mathbf{f} , $x_\omega(V_{s(\omega)}^m) \subseteq V_{t(\omega)}^m$, for all $\omega \in \Omega$.)

The first projection $\pi_\nu : \tilde{\mathcal{F}}_\nu \rightarrow E_{V,\Omega}$ then induced a right derived functor

$$(\pi_\nu)_! : \mathcal{D}_c^b(\tilde{\mathcal{F}}_\nu) \rightarrow \mathcal{D}_c^b(E_{V,\Omega}).$$

Note that π_ν is proper and $\tilde{\mathcal{F}}_\nu$ is smooth. By the Decomposition theorem in [BBD], $(\pi_\nu)_!(\bar{\mathbb{Q}}_l)$ is a semisimple complex in $\mathcal{D}_c^b(E_{V,\Omega})$. Moreover, $(\pi_\nu)_!(\bar{\mathbb{Q}}_l)$ is G_V -equivariant since π_ν is G_V -equivariant.

Let \mathcal{B}_V be the set consisting of all isomorphism classes of simple G_V -equivariant perverse sheaves on $E_{V,\Omega}$ that are in the direct summand of the semisimple complex $(\pi_\nu)_!(\bar{\mathbb{Q}}_l)$ (up to shift) for some $\nu \in \mathcal{X}_{|V|}$. By abuse of language, we say “a complex is in \mathcal{B}_V ” instead of “the isomorphism classes of a complex is in \mathcal{B}_V ”.

Example 4.3. When $|V| = i$, $E_{V,\Omega}$ consists of a single point. The isomorphism classes of the complex $\bar{\mathbb{Q}}_l$ is the only element in \mathcal{B}_V . We denote it by $F_i^{[1]}$.

Let \mathcal{Q}_V be the full subcategory of $\mathcal{D}_c^b(E_{V,\Omega})$ consisting of all complexes on $E_{V,\Omega}$ isomorphic to a direct sum of shifts of finitely many complexes in \mathcal{B}_V .

Let $W \subseteq V$ be an I -graded K -subspace of V and $T = V/W$. For any $x \in E_{V,\omega}$ such that $x_\omega(W_{t(\omega)}) \subseteq W_{h(\omega)}$ for all $\omega \in \Omega$, we call that W is x -stable and it then induces elements x_W and x_T in $E_{W,\Omega}$ and $E_{T,\Omega}$, respectively.

Define E'' to be the variety consisting of all pairs (x, V') , where $x \in E_{V,\Omega}$ and V' is an I -graded subspace of V , such that $|V'| = |W|$ and V' is x -stable.

Define E' to be the variety consisting of all quadruples $(X, V'; R', R'')$ where (x, V') is in E'' , and $R' : V' \rightarrow W$ and $R'' : V/V' \rightarrow T$ are I -graded linear isomorphisms.

Consider the following diagram

$$(*) \quad E_{T,\Omega} \times E_{W,\Omega} \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_{V,\Omega},$$

where the maps are defined as follows.

$p_3 : (x, V') \mapsto x$, $p_2 : (x, V'; R', R'') \mapsto (x, V')$, and $p_1 : (x, V'; R', R'') \mapsto (x', x'')$, where $x'_\omega = R'_{h(\omega)} x_{V'} (R'_{t(\omega)})^{-1}$ and $x''_\omega = R''_{h(\omega)} x_{V/V'} (R''_{t(\omega)})^{-1}$ for all $\omega \in \Omega$.

Note that p_3 is proper, p_2 is a principal $G_T \times G_W$ -bundle and p_1 is smooth with connected fibres.

From $(*)$, we can form a functor

$$(p_3)_!(p_2)_*(p_1)^* : \mathcal{D}_c^b(E_{T,\Omega} \times E_{W,\Omega}) \rightarrow \mathcal{D}_c^b(E_{V,\Omega}).$$

We write $K \star L := (p_3)_!(p_2)_*(p_1)^*(K \boxtimes L)$ for any $K \in \mathcal{D}_c^b(E_{T,\Omega})$ and $L \in \mathcal{D}_c^b(E_{W,\Omega})$.

Lemma 4.4. (1) $K \star L \in \mathcal{Q}_V$ for any $K \in \mathcal{Q}_T$ and $L \in \mathcal{Q}_W$.

(2) $(K \star L) \star M = K \star (L \star M)$ for any $K \in \mathcal{Q}_T$, $L \in \mathcal{Q}_W$ and $M \in \mathcal{Q}_U$.

(3) $(\pi_\nu)_!(\bar{\mathbb{Q}}_l) = (\pi_{(\nu^1)})_!(\bar{\mathbb{Q}}_l) \star \cdots \star (\pi_{(\nu^n)})_!(\bar{\mathbb{Q}}_l)$ where (ν^m) are sequences with only one entry ν^m for $m = 1, \dots, n$.

See [Lus2] for a proof.

By Lemma 4.4 (2), the expression $K \star L \star M$ makes no confusion.

Let $\nu = (\nu_1, \dots, \nu_n)$ be a sequence of elements in $\mathbb{N}[I]$. $((\nu_1, \dots, \nu_n)$ need not satisfy the conditions in Section 4.2 (1).) Let V be an I -graded K -space of dimension $\sum_{m=1}^n \nu_m$. Assume that $V^{(m)}$ ($m = 1, \dots, n$) are I -graded subspaces of V such that $V = \bigoplus_{m=1}^n V^{(m)}$.

Define F'' to be the variety consisting of all pairs (x, V^\bullet) where $x \in E_{V,\Omega}$ and V^\bullet is a flag of type ν such that V^\bullet is x -stable.

Define F' to be the variety consisting of all triples $(x, V^\bullet; \mathbf{g})$ where (x, V^\bullet) is in F'' and \mathbf{g} is a sequence of linear isomorphisms $(g_m : V^{m-1}/V^m \rightarrow V^{(m)} \mid m = 1, \dots, n)$. (Here $V^\bullet = (V = V^0 \supseteq V^1 \supseteq \dots \supseteq V^n = 0)$.)

Consider the following diagram

$$(**) \quad E_{V^{(1)},\Omega} \times \cdots \times E_{V^{(n)},\Omega} \xleftarrow{q_1} F' \xrightarrow{q_2} F'' \xrightarrow{q_3} E_{V,\Omega}$$

where the maps are defined by $q_3 : (x, V^\bullet) \mapsto x$, $q_2 : (x, V^\bullet, \mathbf{g}) \mapsto (x, V^\bullet)$, and $q_1 : (x, V^\bullet, \mathbf{g}) \mapsto (x^{(1)}, \dots, x^{(n)})$ with $x_\omega^{(m)} = (g_m)_{h(\omega)} x_{V^{m-1}/V^m} (g_m)_{t(\omega)}^{-1}$ for $m = 1, \dots, n$.

Similar to p_1, p_2 and p_3 , the morphisms q_1, q_2 and q_3 are smooth with connected fibres, principal $G_{V^{(1)}} \times \cdots \times G_{V^{(n)}}$ -bundle and proper, respectively. Note that when $n = 2$, diagram $(**)$ coincides with diagram $(*)$.

Lemma 4.5. $K^{(1)} \star \cdots \star K^{(n)} = (q_3)_!(q_2)_*(q_1)^*(K^{(1)} \boxtimes \cdots \boxtimes K^{(n)})$ for any $K^{(m)} \in \mathcal{Q}_{V^{(n)}}$ where $m = 1, \dots, n$.

Proof. The statement follows from Lemma 4.4 when $n = 2$. When $n > 2$, the statement can be proved by induction. \square

4.6. Lusztig's algebras. Let $\mathcal{K}_V = \mathcal{K}(\mathcal{Q}_V)$ be the Grothendieck group of the category \mathcal{Q}_V , i.e., it is the abelian group with one generator $\langle L \rangle$ for each isomorphism class of objects in \mathcal{Q}_V with relations: $\langle L \rangle + \langle L' \rangle = \langle L'' \rangle$ if L'' is isomorphic to $L \oplus L'$.

Let v be an indeterminate. Set $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$. Define an \mathbb{A} -module structure on \mathcal{K}_V by $v^n \langle L \rangle = \langle L[n] \rangle$ for any generator $\langle L \rangle \in \mathcal{Q}_V$ and $n \in \mathbb{Z}$. From the construction, it is a free \mathbb{A} -module with basis $\langle L \rangle$ where $\langle L \rangle$ runs over \mathcal{B}_V .

From the construction, we have $\mathcal{K}_V \cong \mathcal{K}_{V'}$, for any V and V' such that $|V| = |V'|$. For each $\nu \in \mathbb{N}[I]$, fix an I -graded vector space V of dimension ν . Let

$$\mathcal{K}_\nu = \mathcal{K}_V, \quad \mathcal{K} = \bigoplus_{\nu \in \mathbb{N}[I]} \mathcal{K}_\nu \quad \text{and} \quad \mathcal{K}_Q = \mathbb{Q}(v) \otimes_{\mathbb{A}} \mathcal{K}.$$

Also let

$$\mathcal{B}_\nu = \mathcal{B}_V \quad \text{and} \quad \mathcal{B}_Q = \cup_{\nu \in \mathbb{N}[I]} \mathcal{B}_\nu.$$

For any $\alpha, \beta \in \mathbb{N}[I]$, the operation \star induces an \mathbb{A} -linear map

$$\star : \mathcal{K}_\alpha \otimes_{\mathbb{A}} \mathcal{K}_\beta \rightarrow \mathcal{K}_{\alpha+\beta}.$$

By adding up these linear maps, we have a linear map

$$\star : \mathcal{K} \otimes_{\mathbb{A}} \mathcal{K} \rightarrow \mathcal{K}.$$

Similarly, the operation \star induces a $\mathbb{Q}(v)$ -linear map

$$\star : \mathcal{K}_Q \otimes_{\mathbb{Q}(v)} \mathcal{K}_Q \rightarrow \mathcal{K}_Q.$$

Proposition 4.7. (1) (\mathcal{K}, \star) (resp. (\mathcal{K}_Q, \star)) is an associative algebra over \mathbb{A} (resp. $\mathbb{Q}(v)$).

(2) \mathcal{B}_Q is an \mathbb{A} -basis of (\mathcal{K}, \star) and a $\mathbb{Q}(v)$ -basis of (\mathcal{K}_Q, \star) .

Proof. The associativity of \star follows from Lemma 4.4 (2). □

Define a new \mathbb{A} -linear map $\circ : \mathcal{K}_\alpha \otimes \mathcal{K}_\beta \rightarrow \mathcal{K}_{\alpha+\beta}$ by

$$x \circ y = v^{m(\alpha, \beta)} x \star y$$

where

$$m(\alpha, \beta) = \sum_{i \in I} \alpha_i \beta_i + \sum_{\omega \in \omega} \alpha_{t(\omega)} \beta_{h(\omega)}.$$

This induces a bilinear map

$$\circ : \mathcal{K} \otimes \mathcal{K} \rightarrow \mathcal{K}.$$

Then

Corollary 4.8. (\mathcal{K}, \circ) is an associative algebra over \mathbb{A} .

The linear map \circ satisfies the associativity due to the fact that $m(-, -)$ is a cocycle. Similarly, we have an associative algebra (\mathcal{K}_Q, \circ) over $\mathbb{Q}(v)$.

4.9. The canonical basis \mathbf{B} of the algebra \mathbf{U}^- . Given any quiver Q , let $c_{ii} = 2$ and

$$c_{ij} = -\#\{\omega \in \Omega \mid \{t(\omega), h(\omega)\} = \{i, j\}\} \quad \text{for } i \neq j.$$

$C = (c_{ij})_{i, j \in I}$ is then a symmetric generalized Cartan matrix. Note that the Cartan matrix C is independent of changes of the orientation of Q .

For any $m \leq n \in \mathbb{N}$, let

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]! = \prod_{m=1}^n [m] \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]! [n-m]!}.$$

Denote by \mathbf{U}^- the negative part of the quantized enveloping algebra attached to the Cartan matrix C . \mathbf{U}^- is the quotient of the free algebra with generators F_i , $i \in I$ by the two-sided ideal generated by

$$(1) \quad \sum_{p=0}^{1-c_{ij}} (-1)^p \begin{bmatrix} 1 - c_{ij} \\ p \end{bmatrix} F_i^p F_j F_i^{1-c_{ij}-p},$$

for $i \neq j \in I$.

Let $F_i^{(n)} = \frac{F_i^n}{[n]!}$ for all $i \in I$ and $n \in \mathbb{N}$. Let ${}_{\mathbb{A}}\mathbf{U}^-$ be the \mathbb{A} -subalgebra of \mathbf{U}^- generated by $F_i^{(n)}$ for $i \in I$ and $n \in \mathbb{N}$. We have

Theorem 4.10. ([Lus2], [Lus4]) *The map $F_i^{(1)} \mapsto F_i^{[1]}$ induces an \mathbb{A} -algebra isomorphism*

$${}_{\mathbb{A}}\phi : {}_{\mathbb{A}}\mathbf{U}^- \rightarrow (\mathcal{K}, \circ)$$

and a $\mathbb{Q}(v)$ -algebra isomorphism

$$\phi : \mathbf{U}^- \rightarrow (\mathcal{K}_Q, \circ).$$

Remark. See [Lus4] for a more general treatment that works for any symmetrisable generalized Cartan matrix C .

Given another quiver $Q' = (I, \Omega, t', h')$ such that

$$\{t(\omega), h(\omega)\} = \{t'(\omega), h'(\omega)\}$$

for all $\omega \in \Omega$. From Theorem 4.10, the map $F_i^{(1)} \mapsto F_i^{[1]}$ induces an $\mathbb{Q}(v)$ -algebra isomorphism

$$\phi' : \mathbf{U}^- \rightarrow (\mathcal{K}_{Q'}, \circ).$$

We have

Theorem 4.11. ([Lus2], [Lus4]) $\phi^{-1}(\mathcal{B}_Q) = (\phi')^{-1}(\mathcal{B}_{Q'})$.

Definition 4.12. $\mathbf{B} = \phi^{-1}(\mathcal{B}_Q)$ is called the Canonical Basis of \mathbf{U}^- .

For each Q , \mathcal{B}_Q gives a presentation of \mathbf{B} . The main goal of this paper is to describe the elements in \mathcal{B}_Q by specifying their supports and the corresponding local systems when Q is affine.

5. THE DESCRIPTION OF THE ELEMENTS IN \mathcal{B}_Q VIA QUIVER REPRESENTATIONS

5.1. Simple equivariant perverse sheaves. Note that elements in \mathcal{B}_Q are isomorphism classes of simple equivariant perverse sheaves. We give a brief description of simple equivariant perverse sheaves.

Let X be an algebraic variety over K with a connected algebraic group G acting on it. Let Y be a smooth, locally closed, irreducible G -invariant subvariety of X and \mathcal{L} an irreducible, G -equivariant, local system on Y . Denote by $j : Y \rightarrow X$ the natural embedding.

Theorem 5.2. ([BBD], [BL]) *The complex*

$$\mathrm{IC}(Y, \mathcal{L}) := j_{!*}(\mathcal{L})[\dim Y]$$

is a simple G -equivariant perverse sheaf on X . Moreover, all simple G -equivariant perverse sheaves on X are of this form.

5.3. Cyclic quivers. When the quiver Q is the cyclic quiver C_p for some $p \in \mathbb{N}$. The description of the elements in \mathcal{B} is given as follows.

Let \mathbb{V} be a $\mathbb{Z}/p\mathbb{Z}$ -graded K -vector space. Recall that a $G_{\mathbb{V}}$ -orbit O in $E_{\mathbb{V}, \Omega_p}$ is aperiodic if for any $x \in O$, the representation (\mathbb{V}, x) is aperiodic (see Section 2.6). Let $\mathcal{O}_{\mathbb{V}}^a$ be the set of all aperiodic $G_{\mathbb{V}}$ -orbits in $E_{\mathbb{V}, \Omega_p}$. Given $O \in \mathcal{O}_{\mathbb{V}}^a$, let $\mathrm{IC}(O, \bar{\mathbb{Q}}_l)$ be the intersection cohomology complex on $E_{\mathbb{V}, \Omega}$ determined by the subvariety O and the constant sheaf $\bar{\mathbb{Q}}_l$ on O . The assignment $O \mapsto \mathrm{IC}(O, \bar{\mathbb{Q}}_l)$ defines a map $\mathcal{O}_{\mathbb{V}}^a \rightarrow \mathcal{B}_{\mathbb{V}}$. Furthermore, we have

Theorem 5.4. ([Lus3, 5.9]) *The map $\mathcal{O}_{\mathbb{V}}^a \rightarrow \mathcal{B}_{\mathbb{V}}$ is bijective.*

5.5. Noncyclic quivers. From now on, we assume that the affine quiver Q is not C_p , for any $p \in \mathbb{N}$. We follow Lusztig's argument in [Lus3, Section 6]. We study three special cases in this section.

First, given $M \in \text{Ind}(Q)$ of dimension vector ν . Assume that M is either preprojective or preinjective. Let V be a K -vector space such that $|V| = \nu$. Let O_M be the G_V -orbit in $E_{V,\Omega}$ corresponding to M . Then we have

Lemma 5.6. ([Lus3, Lemma 6.8]) $\text{IC}(O_M, \bar{\mathbb{Q}}_l) \in \mathcal{B}_V$.

Proof. Since M is either preprojective or preinjective, its self-extension group $\text{Ext}_Q^1(M, M) = 0$, so the corresponding G_V -orbit O_M is open in $E_{V,\Omega}$ (see [CB]). Since $E_{V,\Omega}$ is smooth, $\text{IC}(O_M, \bar{\mathbb{Q}}_l)$ is the constant sheaf $\bar{\mathbb{Q}}_l$ on $E_{V,\Omega}$ up to shift (see [BBD, Lemma 4.3.2]). Now that Q has no oriented cycles, we can order the vertices in I i_1, \dots, i_n ($n = |I|$) in a way such that i_r is a source of the full subquiver Q_r with vertex set $I - \{i_1, \dots, i_{r-1}\}$. For any V of dimension vector ν , let $\boldsymbol{\nu} = (\nu_{i_1} i_1, \dots, \nu_{i_n} i_n)$. By definition, $\mathcal{F}_{\boldsymbol{\nu}}$ consists of a single flag. Also any x in $E_{V,\Omega}$ stabilizes this flag. So the first projection $\pi_{\boldsymbol{\nu}} : \tilde{\mathcal{F}}_{\boldsymbol{\nu}} \rightarrow E_{V,\Omega}$ is an isomorphism. Therefore, we have $(\pi_{\boldsymbol{\nu}})_!(\bar{\mathbb{Q}}_l)[d] = \bar{\mathbb{Q}}_l[d] \in \mathcal{B}_V$. \square

Second, we assume that V is an I -graded K -vector space of dimension vector $q\delta$, where $\delta \in \mathbb{N}[I]$ is the minimal positive imaginary root of the symmetric Euler form (see Section 2.1) associated to Q . We define two varieties as follows.

- (1) The variety $X(0)$. It is the subvariety of $E_{V,\Omega}$ consisting of all elements x such that $(V, x) \simeq R_1 \oplus \dots \oplus R_q$, where R_1, \dots, R_q are pairwise nonisomorphic homogeneous regular simples.
- (2) The variety $\tilde{X}(0)$. This variety consists of all pairs $\{x, (R_1, \dots, R_q)\}$, where $x \in X(0)$ and (R_1, \dots, R_q) is a sequence of representations in $\text{Ind}(Q)$, such that $(V, x) \simeq R_1 \oplus \dots \oplus R_q$.

Note that the dimension vectors of R_m in (1) have to be δ , for all $m \in \{1, \dots, q\}$ and once x is fixed, the set of the representations R_1, \dots, R_q in (2) is completely determined. Note also that the closure of $X(0)$ equals $E_{V,\Omega}$.

In fact, by [R2], we have $\dim X(0) = \dim O_x + q$. Since

$$\dim G_V - \dim E_{V,\Omega} = \langle q\delta, q\delta \rangle = \dim \text{Hom}_Q((V, x), (V, x)) - \dim \text{Ext}^1((V, x), (V, x)),$$

we have

$$\dim E_{V,\Omega} = \dim G_V - \dim \text{Hom}((V, x), (V, x)) + q.$$

So $\dim E_{V,\Omega} = \dim O_x + q = \dim X(0)$. Therefore $E_{V,\Omega}$ is the closure of $X(0)$.

The first projection

$$\pi_1 : \tilde{X}(0) \rightarrow X(0) \quad (x, (R_1, \dots, R_q)) \mapsto x$$

is an S_q -principal covering where S_q is the symmetric group of q letters. S_q acts naturally on $(\pi_1)_*(\bar{\mathbb{Q}}_l)$. Given any irreducible representation χ of S_q , denote by \mathcal{L}_χ the irreducible local system corresponding to the representation χ via the monodromy functor (see [Ive]). Note that \mathcal{L} is a direct summand of $(\pi_1)_*(\bar{\mathbb{Q}}_l)$.

Let $\text{IC}(X(0), \mathcal{L}_\chi)$ be the intersection complex on $E_{V,\Omega}$ determined by $X(0)$ and \mathcal{L}_χ . We then have:

Lemma 5.7. ([Lus3, 6.10 (a)]) $\text{IC}(X(0), \mathcal{L}_\chi) \in \mathcal{B}_V$.

Proof. Let $\boldsymbol{\delta} = (\delta, \dots, \delta)$ such that $|\boldsymbol{\delta}| = q \delta$. Let $\boldsymbol{\pi}_{\boldsymbol{\delta}} : \tilde{\mathcal{F}}_{\boldsymbol{\delta}} \rightarrow E_{V, \Omega}$ be the first projection defined as the morphism $\boldsymbol{\pi}_{\boldsymbol{\nu}}$ in Section 4.2. Given any $x \in X(0)$, let

$$\mathbf{f} = (V = V^0 \supseteq V^1 \supseteq \dots \supseteq V^n = 0)$$

be a flag in $\mathcal{F}_{\boldsymbol{\nu}}$ such that \mathbf{f} is x -stable. Then

- (1) The subrepresentation (V^m, x) is regular, for any $m \in \{0, \dots, n-1\}$.

This is because if $x \in X(0)$, (V, x) is a regular representation. So (V^r, x) can not have preinjective subrepresentations. Now that the dimension vector of (V^r, x) is δ , the defect of (V^r, x) is zero. Thus, (V^r, x) can not have preprojective subrepresentations. Therefore, (V^r, x) is regular.

Fix an element $x \in X(0)$, we decompose $V = \bigoplus_{r=1}^q V(r)$ such that $V(r)$ is x -stable and $|V(r)| = \delta$. By the definition of $X(0)$ and (1), we have $(V, x) \simeq \bigoplus_r (V(r), x)$.

- (2) Moreover, this decomposition is unique up to order.

In fact, if $V = \bigoplus_r W(r)$ is another decomposition, we can reorder the $W(r)$'s such that the subrepresentations $(W(r), x)$ and $(V(r), x)$ are isomorphic, for any r . Fix an isomorphism $f_r : (W(r), x) \rightarrow (V(r), x)$ for each r , then they induce an isomorphism

$$f := \sum_r f_r : (V, x) \rightarrow (V, x)$$

satisfying $f(W(r)) \subseteq V(r)$, for any r . On the other hand, the composition

$$p_{r'} \circ f \circ i_r : V(r) \xrightarrow{i_r} V \xrightarrow{f} V \xrightarrow{p_{r'}} V(r')$$

is naturally a homomorphism of representations in $\text{Hom}_Q((V(r), x), (V(r'), x))$, where $p_{r'}$ and i_r are natural projection and inclusion, respectively. Note that

$$\text{Hom}_Q((V(r), x), (V(r'), x)) = 0, \text{ if } r \neq r'.$$

We have $p_{r'} \circ f \circ i_r = 0$, for $r \neq r'$. So, $f(V(r)) \subseteq V(r)$. But by definition, $f(V(r)) \subseteq W(r)$. Thus, $V(r) = W(r)$. Therefore the decomposition is unique up to order.

From (2), we can define an injective map $\alpha : \tilde{X}(0) \rightarrow \tilde{\mathcal{F}}_{\boldsymbol{\delta}}$ by $\{x, (R_1, \dots, R_q)\} \mapsto (x, \mathbf{f})$, where \mathbf{f} is the flag $(V = V^1 \supseteq V^2 \supseteq \dots \supseteq V^{q+1} = 0)$ such that $V^r = \bigoplus_{k=r}^q V(k)$ and $(V(k), x) \simeq R_k$, for $r = 1, \dots, q$. We then have the following commutative diagram:

$$\begin{array}{ccc} \tilde{X}(0) & \xrightarrow{\pi_1} & X(0) \\ \alpha \downarrow & & \downarrow \\ \tilde{\mathcal{F}}_{\boldsymbol{\delta}} & \xrightarrow{\boldsymbol{\pi}_{\boldsymbol{\delta}}} & E_{V, \Omega} . \end{array}$$

Note that this diagram is Cartesian. So the restriction of $\boldsymbol{\pi}_{\boldsymbol{\delta}}!(\bar{\mathbb{Q}}_l)$ to $X(0)$ is $(\pi_1)_!(\bar{\mathbb{Q}}_l)$. Recall that \mathcal{L}_{χ} is a direct summand of $(\pi_1)_!(\bar{\mathbb{Q}}_l)$ and $X(0)$ is open in $E_{V, \Omega}$. So

- (3) $\text{IC}(X(0), \mathcal{L}_{\chi})$ is a direct summand of $(\boldsymbol{\pi}_{\boldsymbol{\delta}})_!(\bar{\mathbb{Q}}_l)$, up to shift.

By Lemma 4.4 (3),

- (4) $(\boldsymbol{\pi}_{\boldsymbol{\delta}})_!(\bar{\mathbb{Q}}_l) = (\boldsymbol{\pi}_{\boldsymbol{\delta}})_!(\bar{\mathbb{Q}}_l) \star \dots \star (\boldsymbol{\pi}_{\boldsymbol{\delta}})_!(\bar{\mathbb{Q}}_l),$

where δ is regarded as a sequence with only one entry. Observe that $(\pi_\delta)_!(\bar{\mathbb{Q}}_l) = \bar{\mathbb{Q}}_l$. By the proof of Lemma 5.6, they are all in \mathcal{B} (up to shifts). From (3) and (4), $\mathrm{IC}(X(0), \mathcal{L}_\chi) \in \mathcal{B}_V$. Lemma 5.7 is proved. \square

Finally, let T be a tube of period $p \neq 1$. Let $\mathcal{O}_{V,T}^a$ be the set of all aperiodic G_V -orbits O_x in $E_{V,\Omega}$ (see 2.5). Given any $O \in \mathcal{O}_{V,T}^a$, denote by $\mathrm{IC}(O, \bar{\mathbb{Q}}_l)$ the intersection complex on $E_{V,\Omega}$ determined by O and the constant local system $\bar{\mathbb{Q}}_l$ on O . Then, we have

Lemma 5.8. ([Lus3, 6.9 (a)]) $\mathrm{IC}(O, \bar{\mathbb{Q}}_l) \in \mathcal{B}_V$, for any $O \in \mathcal{O}_{V,T}^a$.

Proof. Fix a regular simple R in T , following the construction in Section 3.5, we have an categorical equivalence $F : \mathrm{Rep}(C_p) \rightarrow \mathrm{HT}$, where HT is the full subcategory of $\mathrm{Rep}(Q)$ generated by T and all the homogeneous regular simples such that HT is closed under extensions in $\mathrm{Rep}(Q)$ and taking kernel and cokernels of morphisms in HT . Given an element x in O , there exists a representation $(\mathbb{V}, \theta) \in \mathrm{Rep}(C_p)$ such that $F(\mathbb{V}, \theta) \simeq (V, x)$. In particular, $F(\mathbb{V}) \simeq V$ as K -vector spaces. Consequently, $E_{F(\mathbb{V}),\Omega} \simeq E_{V,\Omega}$. So we can identify $F(\mathbb{V})$ with V and identify the $G_{F(\mathbb{V})}$ -orbit of $F(\theta)$ in $E_{F(\mathbb{V}),\Omega}$ with O in $E_{V,\Omega}$.

Since F is equivalent and O is aperiodic, we have O_θ is aperiodic in $\mathrm{Rep}(C_p)$. By Theorem 5.4, we have

$$(1) \quad \mathrm{IC}(O_\theta, \bar{\mathbb{Q}}_l) \in \mathcal{B}_V.$$

In other words,

$$(2) \quad \mathrm{IC}(O_\theta, \bar{\mathbb{Q}}_l) \text{ is a direct summand of } (\pi_{\mathbf{z}})_!(\bar{\mathbb{Q}}_l), \text{ (up to shift)}$$

for some $\mathbf{z} = (z_1, \dots, z_n)$, $z_s \in \mathbb{Z}/p\mathbb{Z}$.

Define \mathcal{F}'_ν to be the variety consisting of all flags of the form

$$F(f) = (F(\mathbb{V}) \supseteq F(\mathbb{V}^1) \supseteq \dots \supseteq F(\mathbb{V}^n)),$$

where $f = (\mathbb{V} \supseteq \mathbb{V}^1 \supseteq \dots \supseteq \mathbb{V}^n)$ is a flag of type \mathbf{z} .

Define $\tilde{\mathcal{F}}'_\nu$ to be the variety consisting of all pair (x, \mathbf{f}) , where $x \in E_1$ and $\mathbf{f} \in \mathcal{F}'_\nu$, such that \mathbf{f} is x -stable.

Define $\tilde{\mathcal{F}}''_\nu$ to be the variety consisting of all pairs (x, \mathbf{f}) , where $x \in E_2$ and $\mathbf{f} \in \mathcal{F}_\nu$ such that \mathbf{f} is x -stable. Then we have $\tilde{\mathcal{F}}''_\nu = G_V \times^{H_V} \tilde{\mathcal{F}}_\nu$.

Consider the following commutative diagram

$$\begin{array}{ccccccc} \tilde{\mathcal{F}}_{\mathbf{z}} & \xrightarrow{\tilde{F}} & \tilde{\mathcal{F}}'_\nu & \xrightarrow{\tilde{i}_1} & \tilde{\mathcal{F}}''_\nu & \xrightarrow{\tilde{i}} & \tilde{\mathcal{F}}_\nu \\ \pi_{\mathbf{z}} \downarrow & & \pi_{\nu'} \downarrow & & \pi_{\nu''} \downarrow & & \pi_\nu \downarrow \\ E_{\mathbb{V},\Omega_p} & \xrightarrow{F} & E_1 & \xrightarrow{i_1} & E_2 & \xrightarrow{i} & E_{V,\Omega}, \end{array}$$

where the vertical maps are first projections, i, i_1, \tilde{i} and \tilde{i}_1 are inclusions, and $\tilde{F} : (\theta, f) \mapsto (F(\theta), F(\mathbf{f}))$. Note that all squares are Cartesians.

Let O' be the H_V -orbit of $F(\theta)$ in E_1 . Denote by $\mathrm{IC}(O', \bar{\mathbb{Q}}_l)$ the intersection complex on E_1 determined by O' and the trivial local system $\bar{\mathbb{Q}}_l$.

From Section 3.5 (e), the statement (2) and the Cartesian square on the left in the diagram above, we have

$$(3) \quad \mathrm{IC}(O', \bar{\mathbb{Q}}_l) \text{ is a direct summand of } (\pi'_{\nu})_!(\bar{\mathbb{Q}}_l)[d'], \text{ for some } d'.$$

Let O'' be the G_V -orbit of $F(\theta)$ in E_2 . (In fact, $O'' = O$.) Denote by $\mathrm{IC}(O'', \bar{\mathbb{Q}}_l)$ be the intersection complex on E_2 determined by O'' and $\bar{\mathbb{Q}}_l$.

By Section 3.5 (i) and [BL, Theorem 2.6.3], the derived functor

$$i_1^* : \mathcal{D}_{G_V}(E_2) \rightarrow \mathcal{D}_{H_V}(E_1)$$

is a categorical equivalence. In particular,

$$(4) \quad i_1^*(\mathrm{IC}(O'', \bar{\mathbb{Q}}_l)) = \mathrm{IC}(O', \bar{\mathbb{Q}}_l).$$

Since the middle square in the above diagram is Cartesian and π_ν'' is proper, by the base change Theorem for proper morphism ([BBD, Theorem 6.2.5]), We have

$$(5) \quad i_1^*(\pi_\nu'')_!(\bar{\mathbb{Q}}_l) = (\pi_\nu')_!(\bar{\mathbb{Q}}_l).$$

Thus by (3), (4) and (5), we have

$$(6) \quad \mathrm{IC}(O'', \bar{\mathbb{Q}}_l) \text{ is a direct summand of } (\pi_\nu'')_!(\bar{\mathbb{Q}}_l)[d''], \text{ for some } d''.$$

The right square is Cartesian, so we have

$$(7) \quad i^*(\pi_\nu)_!(\bar{\mathbb{Q}}_l) = (\pi_\nu'')_!(\bar{\mathbb{Q}}_l).$$

Note that the closure of E_2 is $\pi_\nu(\tilde{\mathcal{F}}_\nu)$ and E_2 is open in its closure. By (6) and (7), we have

$$(8) \quad \mathrm{IC}(O, \bar{\mathbb{Q}}_l) \text{ is a direct summand of } (\pi_\nu)_!(\bar{\mathbb{Q}}_l)[d], \text{ for some } d.$$

But $(\pi_\nu)_!(\bar{\mathbb{Q}}_l)$ is a direct sum of simple perverse sheaves from \mathcal{B} with shifts. Therefore, $\mathrm{IC}(O, \bar{\mathbb{Q}}_l) \in \mathcal{B}_V$. Lemma 5.8 follows. \square

5.9. General Cases. In this section, we study general cases.

Recall that $\mathrm{Ind}(Q)$ is the set of representatives of pairwise nonisomorphic indecomposable representations of Q .

Given any $\nu \in \mathbb{N}[I]$, denote by Δ_ν the set of all pairs (σ, λ) where $\sigma : \mathrm{Ind}(Q) \rightarrow \mathbb{N}$ is a function and λ is the sequence (0) or a sequence $(\lambda_1, \dots, \lambda_n)$ of decreasing positive integers satisfying the following properties:

- (a) $\prod_{m=0}^{r-1} \sigma((\Phi^+)^m(R)) = 0$ for any regular representation $R \in \mathrm{Ind}(Q)$ of period r ;
- (b) $\sum_{M \in \mathrm{Ind}(Q)} \sigma(M)|M| + \sum_{m=1}^n \lambda_m \delta = \nu$ if $\lambda = (\lambda_1, \dots, \lambda_n)$;
- (c) $\sum_{M \in \mathrm{Ind}(Q)} \sigma(M)|M| = \nu$ if $\lambda = (0)$.

From (a), if R is homogeneous, $\sigma(R) = 0$. From (b) and (c), the function σ has finite support.

Given any $(\sigma, \lambda) \in \Delta_\nu$, fix a K -vector space V of dimension vector ν . Define the varieties $X(\sigma, \lambda)$ and $\tilde{X}(\sigma, \lambda)$, the map π_1 and the irreducible local system \mathcal{L}_λ as follows.

If $\lambda = (\lambda_1, \dots, \lambda_n)$, $X(\sigma, \lambda)$ is the subvariety of $E_{V, \Omega}$ consisting of all elements x such that

$$(V, x) \simeq \oplus_{M \in \mathrm{Ind}(Q)} M^{\sigma(M)} \oplus R_1 \oplus \dots \oplus R_q,$$

where $M^{\sigma(M)}$ is the direct sum of $\sigma(M)$ copies of M , R_1, \dots , and R_q are pairwise nonisomorphic homogeneous regular simples. The variety $\tilde{X}(\sigma, \lambda)$ is the variety consisting of all pairs

$$(x, (R_1, \dots, R_q))$$

where $x \in X(\sigma, \lambda)$ and (R_1, \dots, R_q) is a sequence of homogeneous regular simples in $\text{Ind}(Q)$ completely determined by x up to order. The map $\pi_1 : \tilde{X}(\sigma, \lambda) \rightarrow X(\sigma, \lambda)$ is the first projection. Note that the first projection $\pi_1 : \tilde{X}(\sigma, \lambda) \rightarrow X(\sigma, \lambda)$ is a S_q -principal covering. The sequence λ determines an irreducible representation $\chi(\lambda)$ of the symmetric group S_q . Define \mathcal{L}_λ to be the direct summand of $(\pi_1)_*(\bar{\mathbb{Q}}_l)$ corresponding to the irreducible representation of S_q determined by the partition λ .

If $\lambda = (0)$, the variety $X(\sigma, \lambda)$ is the subvariety of $E_{V,\Omega}$ consisting of all elements s such that $(V, x) \simeq \oplus_{M \in \text{Ind}(Q)} M^{\sigma(M)}$. $\tilde{X}(\sigma, \lambda)$ is $X(\sigma, \lambda)$. π_1 is the identity map $\tilde{X}(\sigma, \lambda) \rightarrow X(\sigma, \lambda)$. Denote by \mathcal{L}_λ the trivial local system on $X(\sigma, \lambda)$.

Note that when $\lambda = (0)$, the variety $X(\sigma, \lambda)$ is a G_V -orbit in $E_{V,\Omega}$.

Let $\text{IC}(\sigma, \lambda) = \text{IC}(X(\sigma, \lambda), \mathcal{L}_\lambda)$ be the simple perverse sheaf on $E_{V,\Omega}$ determined by $X(\sigma, \lambda)$ and \mathcal{L}_λ .

Proposition 5.10. ([Lus3, Proposition 6.7]) $\text{IC}(\sigma, \lambda) \in \mathcal{B}_V$.

The proof will be given in the next section.

By Proposition 5.10, the assignment $(\sigma, \lambda) \mapsto \text{IC}(\sigma, \lambda)$ defines a map $\Delta_\nu \rightarrow \mathcal{B}_V$. This map is injective due to the fact that different pairs (σ, λ) determine different perverse sheaves. Moreover, the cardinalities $|\Delta_\nu| \stackrel{(1)}{=} |\text{Irr } \Lambda_V| \stackrel{(2)}{=} |\mathcal{P}_V|$, where $\text{Irr } \Lambda_V$ is the set of all irreducible component of the variety Λ_V constructed in [Lus2]. The equality (1) holds by [R2, corollary 5.3] and the equality (2) holds by [Lus3, Theorem 4.16 (b)]. By definitions, the two sets are of finite order. Therefore we have

Theorem 5.11. ([Lus3, Theorem 6.16 (b)]) *The map $\Delta_\nu \rightarrow \mathcal{B}_V$ is bijective.*

5.12. Proof of Proposition 5.10. We preserve the setting of Section 5.9. Given any element $(\sigma, \lambda) \in \Delta_\nu$, recall that the variety $X(\sigma, \lambda)$ contains all elements $x \in E_{V,\Omega}$ such that

$$(V, x) \simeq \oplus M^{\sigma(M)} \oplus (R_1 \oplus \dots \oplus R_q),$$

where R_1, \dots, R_q are pairwise nonisomorphic homogeneous regular simple representations in $\text{Ind}(Q)$.

We can write the representation $\oplus M^{\sigma(M)} \oplus (R_1 \oplus \dots \oplus R_q)$ as

$$O_1 \oplus \dots \oplus O_n,$$

such that

$$(a) \quad \text{Ext}^1(O_m, O_{m'}) = 0 \text{ and } \text{Hom}(O_{m'}, O_m) = 0, \text{ if } m < m',$$

and O_m has one of the following forms:

- (1) $O_m = M^{\sigma(M)}$ where $M \in \text{Ind}(Q)$ is preprojective or preinjective;
- (2) $O_m = \oplus_{M \in T} M^{\sigma(M)}$, where T is a tube;
- (3) $O_m = R_1 \oplus \dots \oplus R_q$.

The condition (a) can be accomplished by putting the preprojective (resp. preinjective) O_m 's in case (1) in the first (resp. last) part of the sequence and putting the O_m 's in cases (2) and (3) in the middle part of the sequence, then adjusting the O_m 's in case (1) such that they satisfy the condition (a). (This can be done due to Lemma 2.4.) Note that any order of the O_m 's in case (2) and (3) already satisfies the condition (a).

For each m , let $\nu_m = |O_m|$. By the definition of O_m , this is well-defined. Fix a K -vector space $V(m)$ such that $|V(m)| = \nu_m$. By abuse of notations, denote by O_m the subvariety in $E_{V(m),\Omega}$ consisting of all elements x such that $(V, x) \simeq O_m$. (Note that O_m in case (3) is nothing but $X(0)$ in $E_{V(m),\Omega}$ in Section 5.5.)

Define the irreducible local system \mathcal{L}_m on O_m by

- $\mathcal{L}_m = \bar{\mathbb{Q}}_l$ when O_m is case (1) or (2);
- $\mathcal{L}_m = \mathcal{L}_{\chi(\lambda)}$ (see Lemma 5.7) when O_m is case (3).

Denote by $\mathrm{IC}(O_m, \mathcal{L}_m)$ the intersection complex on $E_{V(m),\Omega}$ determined by O_m and \mathcal{L}_m . Then $\mathrm{IC}(O_m, \mathcal{L}_m)$ is in $\mathcal{P}_{V(m)}$ by Lemma 5.6, 5.8 and 5.7 for O_m in the case (1), (2) and (3), respectively. So the semisimple complex

$$\mathrm{IC}(O_1, \mathcal{L}_1) \star \cdots \star \mathrm{IC}(O_n, \mathcal{L}_n)$$

on $E_{V,\Omega}$ is in \mathcal{Q}_V (see Section 4.2).

To prove Proposition 5.10, it suffices to show that $\mathrm{IC}(\sigma, \lambda)$ is a direct summand of the semisimple complex $\mathrm{IC}(O_1, \mathcal{L}_1) \star \cdots \star \mathrm{IC}(O_n, \mathcal{L}_n)$ up to shift.

For simplicity, denote by E_m the variety $E_{V(m),\Omega}$ for $m = 1, \dots, n$. Let \bar{O}_m the closure of O_m in E_m for $m = 1, \dots, n$. Recall from Section 4.2, we have the following diagram

$$(**) \quad E_1 \times \cdots \times E_n \xleftarrow{q_1} F' \xrightarrow{q_2} F'' \xrightarrow{q_3} E_{V,\Omega}.$$

By Lemma 4.5, we have

$$(4) \quad \mathrm{IC}(O_1, \mathcal{L}_1) \star \cdots \star \mathrm{IC}(O_n, \mathcal{L}_n) = (q_3)_!(q_2)_*(q_1)^*(\mathrm{IC}(O_1, \mathcal{L}_1) \boxtimes \cdots \boxtimes \mathrm{IC}(O_n, \mathcal{L}_n)).$$

Let $\tilde{\mathcal{F}}''$ be the subvariety of F'' consisting of all elements (x, V^\bullet) such that the induced representations $(V^{m-1}/V^m, x)$ is in \bar{O}_m for $m = 1, \dots, n$.

Let $\tilde{\mathcal{F}}'$ be the subvariety of F' consisting of all elements (x, V^\bullet) such that the induced representations $(V^{m-1}/V^m, x)$ is in O_m for any $m = 1, \dots, n$.

Denote by A'' the subvariety of F' consisting of all triples $(x, V^\bullet, \mathbf{g})$ in E' such that the induced representations of x are in the \bar{O}_m 's.

Denote by A' the subvariety of F' consisting of all triples $(x, V^\bullet, \mathbf{g})$ such that the induced representations of x are in the O_m 's.

Consider the following commutative diagram:

$$\begin{array}{ccccc} O_1 \times \cdots \times O_n & \xleftarrow{q_1''} & A' & \xrightarrow{q_2''} & \tilde{\mathcal{F}}' \\ j' \downarrow & & i' \downarrow & & i \downarrow \\ \bar{O}_1 \times \cdots \times \bar{O}_n & \xleftarrow{q_1'} & A'' & \xrightarrow{q_2'} & \tilde{\mathcal{F}}'' \\ j \downarrow & & \rho' \downarrow & & \rho \downarrow \\ E_1 \times \cdots \times E_n & \xleftarrow{q_1} & F' & \xrightarrow{q_2} & F'' \xrightarrow{q_3} E_{V,\Omega} \end{array}$$

where the bottom row is the diagram (**), q_1' and q_1'' are the restrictions of q_1 , q_2' and q_2'' are the restrictions of q_2 , and the vertical maps are natural embeddings.

From the definitions, the squares in the above diagram are Cartesian. Since q_3 and $\pi := q_3\rho$ are proper, ρ is proper. Hence ρ' is proper. By the base change theorem for proper morphisms, we have

$$q_1^* j_! = \rho_!(q_1')^* \quad \text{and} \quad q_2^* \rho_! = \rho_!(q_2')^*.$$

Note that

$$\boxtimes_{m=1}^n \mathrm{IC}(O_m, \mathcal{L}_m) = j_{!*} j'_{!*}(\boxtimes \mathcal{L}_m)[\dim O_1 \times \cdots \times O_n].$$

Set $d = \dim O_1 \times \cdots \times O_n$. Note that $j_{!*} = j_!$. So

$$\begin{aligned} (q_3)_!(q_2)_b(q_1)^*(\boxtimes_{m=1}^n \mathrm{IC}(O_m, \mathcal{L}_m)) &= (q_3)_!(q_2)_b(q_1)^* j_{!*} j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d] \\ &= (q_3)_!(q_2)_b \rho_!(q'_1)^*(j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d]) = (q_3)_! \rho_!(q'_2)_b(q'_1)^* j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d] \\ &= \pi_!(q'_2)_b(q'_1)^* j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d]. \end{aligned}$$

Denote by \mathcal{L} the complex $(q'_2)_b(q'_1)^* j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d]$. So (4) becomes

$$(5) \quad (q_3)_!(q_2)_b(q_1)^*(\boxtimes_{m=1}^n \mathrm{IC}(O_m, \mathcal{L}_m)) = \pi_!(\mathcal{L}).$$

Note that $j_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d]$ is a simple perverse sheaf. Recall that q'_1 is smooth with connected fibres, by [BBD, Proposition 4.2.5],

$$(q'_1)^*[d_1](j_{!*}(\boxtimes_{m=1}^n (\mathcal{L}_m)))[d] \quad \text{is a simple perverse sheaf on } A''.$$

Since q'_2 is a principal $G_{V(1)} \times \cdots \times G_{V(n)}$ -bundle, \mathcal{L} is a simple perverse sheaf on $\tilde{\mathcal{F}}''$ up to shift. Note that $O_1 \times \cdots \times O_n$ is a smooth variety, by [BBD, Lemma 4.3.2],

$$(j')^* j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m) = \boxtimes_{m=1}^n \mathcal{L}_m.$$

Since the top square in the above diagram are Cartesian,

$$(6) \quad i^* \mathcal{L} = (q'_2)_b(q'_1)^*(j')^* j'_{!*}(\boxtimes_{m=1}^n \mathcal{L}_m)[d] = (q'_2)_b(q'_1)^*(\boxtimes_{m=1}^n \mathcal{L}_m)[d].$$

We set $X = X(\sigma, \lambda) - X(\sigma, \lambda) \cap \pi(\tilde{\mathcal{F}}'' - \tilde{\mathcal{F}}')$ and $Y = \pi^{-1}(X)$. Then

X is open dense in $X(\sigma, \lambda)$ and the restriction $\pi^0 : Y \rightarrow X$ is an isomorphism.

(See [L2, 6.12 (a) (b)] or [Li, 5.6], this is where the condition (a) is used.) Thus we have the following diagram:

$$O_1 \times \cdots \times O_n \xleftarrow{q_1^0} F^0 \xrightarrow{q_2^0} Y \xrightarrow{\pi^0} X,$$

where $F^0 = (q'_2)^{-1}(Y)$ and q_1^0, q_2^0, π^0 are the natural restrictions of q'_1, q'_2 and π , respectively. By the definition of $\mathcal{L}_{\chi(\lambda)}$, we have

$$(\pi^0 q_2^0)^*(\mathcal{L}_{\chi(\lambda)}|_X) = (q_1^0)^*(\boxtimes_m \mathcal{L}_m).$$

Also by (6), $(q_1^0)^*(\boxtimes_m \mathcal{L}_m) = (q_2^0)^* \mathcal{L}|_Y$. So $(\pi^0)^*(\mathcal{L}_{\chi(\lambda)}|_X) = \mathcal{L}|_Y$. Since $\pi^0 : Y \rightarrow X$ is an isomorphism,

$$(\pi)_!(\mathcal{L})[-d]|_X \simeq \mathcal{L}_{\chi(\lambda)}|_X.$$

Therefore, the intersection complex $\mathrm{IC}(X, \mathcal{L}_{\chi(\lambda)}|_X)$ is a direct summand of $\pi_!(\mathcal{L})[d]$. Since X is open dense in the closure of $X(\sigma, \lambda)$, $\mathrm{IC}(X, \mathcal{L}_{\chi(\lambda)}|_X) = \mathrm{IC}(\sigma, \lambda)$. Proposition 5.10 follows.

6. COMMENTS

Note that we deal with the characterizations of the canonical bases in the symmetric cases. It may be of interest to characterize the canonical bases in the nonsymmetric cases.

From the proof of Proposition 5.10, the set

$$\{\mathrm{IC}(O_1, \mathcal{L}_1) \star \cdots \star \mathrm{IC}(O_n, \mathcal{L}_n) \mid (\sigma, \lambda) \in \Delta_\nu, \nu \in \mathbb{N}[I]\}$$

is a $\mathbb{Q}(v)$ -basis of the algebra (\mathcal{K}_Q, \star) (see 4.6). Moreover by looking closer to the shifts, one can show that

Corollary 6.1. *The set*

$$\{\mathrm{IC}(O_1, \mathcal{L}_1) \circ \cdots \circ \mathrm{IC}(O_n, \mathcal{L}_n) \mid (\sigma, \lambda) \in \Delta_\nu, \nu \in \mathbb{N}[I]\}$$

is an \mathbb{A} -basis of the algebra $(\mathcal{K}, \circ) = {}_{\mathbb{A}}\mathbf{U}^-$ (see 4.6) and stable under bar involution. The transition matrix between this basis and the canonical basis is upper triangular with entries in the diagonal equal 1 and entries above the diagonal in \mathbb{A} .

Proof. For simplicity, we write $\mathbf{C}_{\sigma, \lambda}$ for the complex

$$\mathrm{IC}(O_1, \mathcal{L}_1) \circ \cdots \circ \mathrm{IC}(O_n, \mathcal{L}_n).$$

Since the bar involution commutes with the multiplication \circ (see [Lus4]) and the intersection cohomology complexes are self-dual, the complex $\mathbf{C}_{\sigma, \lambda}$ is stable under the bar involution. From the proof of Proposition 5.10, we see that

$$\mathbf{C}_{\sigma, \lambda} = \mathrm{IC}(\sigma, \lambda)[d] \oplus P,$$

for some d and $\mathrm{supp}(P) \subseteq \overline{X(\sigma, \lambda)} - X(\sigma, \lambda)$. But $\mathbf{C}_{\sigma, \lambda}$ is bar invariant, d has to be zero. Corollary follows. \square

The relationship between this basis and the “canonical basis” defined in [LXZ] deserves further investigation.

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